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ORIGINAL PAPER

Sharp sufficient conditions for mean convergence of the maximal partial sums of dependent random variables with general norming sequences

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Abstract
This paper provides sharp sufficient conditions for mean convergence of the maximal partial sums from triangular arrays of dependent random variables with general norming sequences. As an application, we use this result to give a positive answer to an open question in [Test 32(1):74–106, 2023] concerning mean convergence for the maximal partial sums under regularly varying moment conditions. The techniques developed in the present work also enable us to establish a result on mean convergence for sums of pairwise negatively dependent random variables, which gives an improvement of the main result of Sung [Appl Math Lett 26(1):18–24, 2013] and Ordóñez Cabrera and Volodin [J Math Anal Appl 305(2):644–658, 2005].

Keywords Mean convergence · Dependent random variables · Maximal partial sum · Regularly varying moment condition · Regularly varying norming sequence

Mathematics Subject Classification 60F25

1 Introduction

In 1971, Chow [7] established a mean convergence theorem under a uniform integrability condition. A special case of Chow's result reads as follows.

Theorem 1.1 (Chow [7]) *Let $1 \le p < 2$ and let $\{X_n, n \ge 1\}$ be a sequence of independent random variables such that the sequence $\{|X_n|^p, n \ge 1\}$ is uniformly integrable. Then*

$$\frac{1}{n^{1/p}} \sum_{i=1}^{n} (X_i - \mathbb{E}X_i) \stackrel{\mathcal{L}_p}{\to} 0 \text{ as } n \to \infty.$$

Theorem 1.1 was extended by Ordóñez Cabrera and Volodin [13], Lita da Silva [14], Sung [15], and Wu and Guan [22] to the case where the underlying random variables are pairwise

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negatively dependent and satisfy some general uniform integrability conditions. Recently, the authors in [2, 8, 18] investigated laws of large numbers where the norming sequence is a regularly varying function, instead of the classical power function. A real-valued function $L(\cdot)$ is said to be *slowly varying* if it is a positive and measurable function on $[A, \infty)$ for some $A \geq 0$, and for each $\lambda > 0$, $\lim_{x\to\infty} L(\lambda x)/L(x) = 1$. A function $R(\cdot)$ is said to be *regularly varying with the index of regular variation* ρ if it can be written in the form $R(x) = x^\rho L(x)$, where $L(\cdot)$ is a slowly varying function. For a slowly varying function $L(\cdot)$, there exists a slowly varying function $\tilde{L}(\cdot)$, unique up to asymptotic equivalence, satisfying

$$\lim_{x\to\infty} L(x)\tilde{L}\left(xL(x)\right) = 1 \text{ and } \lim_{x\to\infty} \tilde{L}(x)L\left(x\tilde{L}(x)\right) = 1. \tag{1.1}$$

The function $\tilde{L}$ is called the *de Bruijn conjugate* of $L(\cdot)$ (see Theorem 1.5.13 in Bingham et al. [4]). If $L(x) = \log^\gamma(x)$ or $L(x) = \log^\gamma(\log(x))$ for some $\gamma \in \mathbb{R}$, then $\tilde{L}(x) = 1/L(x)$. Especially, if $L(x) \equiv 1$, then $\tilde{L}(x) \equiv 1$. On the important role of regularly varying functions in probability and mathematical analysis, we refer to Bingham et al. [4], Jessen and Mikosch [11].

The following theorem was proved in [18, Corollary 4.10].

Theorem 1.2 *Let* $1 \leq p < 2$ *and let* $\{X_n, n \geq 1\}$ *be a sequence of independent integrable random variables. Let* $L(\cdot)$ *be an increasing slowly varying function defined on* $[0, \infty)$ *and let* $\tilde{L}(\cdot)$ *be the de Bruijn conjugate of* $L(\cdot)$. *If the sequence* $\{|X_n|^p L(|X_n|^p), n \geq 1\}$ *is uniformly integrable in the Cesàro sense, then*

$$\frac{1}{n^{1/p}\tilde{L}^{1/p}(n)} \max_{1\leq k\leq n} \left|\sum_{i=1}^{k} (X_i - \mathbb{E}(X_i))\right| \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty. \tag{1.2}$$

Theorems 1.1 and 1.2 lead to a natural question as follows.

Question 1 Under the assumptions of Theorem 1.2, does convergence in mean of order p prevail in the conclusion (1.2)?

This question was raised as an open problem in [18, Section 5]. In order to address the problem, we may consider applying the result in [13, 15, 17, 22], which establishes mean convergence for the partial sums with general norming sequence $b_n, n \geq 1$. However, these papers assumed a condition of the form

$$\sup_{n\geq 1} \frac{1}{b_n^p} \sum_{i=1}^{n} \mathbb{E}|X_i|^p < \infty, \tag{1.3}$$

which may not hold for the case where $\sup_{i\geq 1} \mathbb{E}|X_i|^p < \infty$ and $b_n \equiv n^{1/p}\tilde{L}^{1/p}(n)$ since $\tilde{L}^{1/p}(n)$ may approach zero. Here, again, $L(\cdot)$ is an increasing slowly varying function and $\tilde{L}(\cdot)$ is the de Bruijn conjugate of $L(\cdot)$. As noted in [18], the existing methods do not appear to be effective in this situation.

The initial objective of this paper is to give a positive answer to Question 1. We will replace condition (1.3) by a weaker condition that

$$\sup_{n\geq 1} \frac{1}{b_n^p} \sum_{i=1}^{n} \mathbb{E}\left(|X_i|^p \mathbf{1}\left(|X_i| > b_n\varepsilon\right)\right) \leq \frac{C_1}{\varepsilon^\delta} \text{ for all } 0 < \varepsilon < 1, \tag{1.4}$$

where $C_1 > 0$ and $\delta \in (0, 2-p)$ do not depend on ε, and then we prove a mean theorem for arrays of dependent random variables with general norming constants under (1.4).

It is clear that (1.3) implies (1.4) since

$$\sup_{n\ge 1}\frac{1}{b_n^p}\sum_{i=1}^{n}\mathbb{E}\left(|X_i|^p\mathbf{1}\left(|X_i|>b_n\varepsilon\right)\right)\le \sup_{n\ge 1}\frac{1}{b_n^p}\sum_{i=1}^{n}\mathbb{E}|X_i|^p$$
$$\le\left(\sup_{n\ge 1}\frac{1}{b_n^p}\sum_{i=1}^{n}\mathbb{E}|X_i|^p\right)\frac{1}{\varepsilon^{\delta}}$$

for all $0<\varepsilon<1$ and $\delta>0$. It is also not hard to see that there are examples where (1.4) holds but (1.3) fails (see, e.g., Example 5.3 in Sect. 5). We will prove in Sect. 4 that if the sequence $\{|X_n|^pL^p(|X_n|), n\ge 1\}$ is uniformly integrable, then (1.4) is fulfilled with $b_n\equiv n^{1/p}\tilde{L}^{1/p}(n)$.

Our method also enables us to give an improvement of Theorem 2.1 of Sung [15] concerning mean convergence for the partial sums of pairwise negatively dependent random variables and, as a result, improves Theorems 3.1−3.3 of Wu and Guan [22] and Theorem 1 of Ordóñez Cabrera and Volodin [13]. It is worth mentioning that our proof is totally different from that of Sung [15], Wu and Guan [22], and Ordóñez Cabrera and Volodin [13].

Throughout the paper, $\{k_n, n\ge 1\}$ denotes a sequence of positive integers. The de Bruijn conjugate of a slowly varying function $L(\cdot)$ is always denoted by $\tilde{L}(\cdot)$. The symbols $C_0, C_1, c_0, \ldots$ denote positive universal constants which may not be necessarily the same in each appearance, and $\mathbf{1}(A)$ denotes the indicator function of the set A. For a real number x, $\log x$ denotes the natural logarithm (base e) of $\max\{x, \mathrm{e}\}$, and $\lfloor x\rfloor$ denotes the greatest integer that is smaller than or equal to x.

The rest of the paper is organized as follows. Some preliminaries used in proving the main theorems are consolidated into Sect. 2. Section 3 focuses on a mean convergence theorem for the maximal partial sums with general norming sequences. In Sect. 4, we apply the result in Sect. 3 to provide a positive answer to Question 1. Section 5 presents an improvement of Theorem 1 in [13], Theorem 2.1 in [15], and Theorems 3.1–3.3 in [22].

2 Preliminaries

In this section, notation, technical definitions, and lemmas which are needed in connection with the main results will be presented.

A collection $\{X_i, 1\le i\le n\}$ of random variables is said to satisfy *Condition* (M_2) if for any collection of increasing functions $\{f_i, 1\le i\le n\}$ with $\mathbb{E}f_i(X_i)=0$, $1\le i\le n$, there exists a constant C_0 such that

$$\mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}f_i\left(X_i\right)\right|\right)^2\le C_0\sum_{i=1}^{n}\mathbb{E}\left(f_i(X_i)\right)^2, \tag{2.1}$$

provided the expectations are finite.

Condition (M_2) holds for independent sequences and many other dependent structures, including martingale differences, negatively associated sequences, and ρ^*-mixing sequences with $\rho_1^*<1$. We also consider another weaker dependence structure defined as follows. A collection $\{X_i, 1\le i\le n\}$ of random variables is said to satisfy *Condition* (H_2) if for all increasing function $\{f_i, 1\le i\le n\}$ with $\mathbb{E}f_i(X_i)=0$, $1\le i\le n$, there exists a constant C_0 such that

$$\mathbb{E}\left(\sum_{i=1}^{n} f_i\left(X_i\right)\right)^2 \le C_0 \sum_{i=1}^{n} \mathbb{E}\left(f_i(X_i)\right)^2, \tag{2.2}$$

provided the expectations are finite. Condition (H_2) holds for pairwise independent sequences, pairwise negatively dependent sequences, and extended negatively dependent sequences. Of course, if (M_2) is satisfied, then so is (H_2). Various limit theorems under these dependence structures were recently studied by some authors. We refer to [1, 3, 17–21, 24] and references therein.

A triangular array $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ of random variables is said to be *uniformly integrable in the Cesàro sense* (see Chandra [5]) if

$$\lim_{a\to\infty} \frac{1}{k_n} \sum_{i=1}^{k_n} \mathbb{E}(|X_{n,i}|\mathbf{1}(|X_{n,i}| > a)) = 0.$$

In [6], Chandra and Goswami presented the de La Vallée–Poussin criterion for uniform integrability in the Cesàro sense which reads as follows: A triangular array of random variables $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ is uniformly integrable in the Cesàro sense if and only if there exists a measurable function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$, $g(x)/x \to \infty$ as $x \to \infty$, and

$$\sup_{n\ge 1} \frac{1}{k_n} \sum_{i=1}^{k_n} \mathbb{E}(g(|X_{n,i}|)) < \infty.$$

The following two lemmas are used in the proof of the main results. The first lemma provides a necessary condition for a weak law of large numbers for the maximum $\max_{1\le i\le k_n} |X_{n,i}| \xrightarrow{\mathbb{P}} 0$. The proof follows from Proposition 2.5 of Thành [21] and so is omitted. We also refer to Lemma 3.2 of Wu and Wang [23] for a similar result.

Lemma 2.1 *Let $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ be a triangular array of random variables such that for all $n \ge 1$, the collection $\{X_{n,i}, 1 \le i \le k_n\}$ satisfies Condition (H_2). Then there exists a constant C_1 depending only on C_0 such that for all $\varepsilon > 0$ and $n \ge 1$,*

$$\left(1 - \mathbb{P}\left(\max_{1\le i\le k_n} |X_{n,i}| > \varepsilon\right)\right)^2 \sum_{i=1}^{k_n} \mathbb{P}\left(|X_{n,i}| > \varepsilon\right) \le C_1 \mathbb{P}\left(\max_{1\le i\le k_n} |X_{n,i}| > \varepsilon\right). \tag{2.3}$$

Furthermore, if $\max_{1\le i\le k_n} |X_{n,i}| \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$, then for all $\varepsilon > 0$, there exists n_0 depending only on ε such that

$$\sum_{i=1}^{k_n} \mathbb{P}\left(|X_{n,i}| > x\right) \le 4C_1 \mathbb{P}\left(\max_{1\le i\le k_n} |X_{n,i}| > x\right) \text{ for all } x \ge \varepsilon, n \ge n_0,$$

where C_1 is as in (2.3), and therefore

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} \mathbb{P}\left(|X_{n,i}| > \varepsilon\right) = 0 \text{ for all } \varepsilon > 0.$$

The last lemma is Corollary 3.1.1 in [9].

Lemma 2.2 *Let $\{X_i, 1 \le i \le n\}$ be independent mean 0 random variables such that for some constant $A > 0$,*

$$\max_{1 \le i \le n} |X_i| \le A \ \ a.s.$$

Then for all $\varepsilon > 0$, we have

$$\sum_{i=1}^{n} \mathbb{E}X_i^2 \le \varepsilon^2 + \frac{(\varepsilon + A)^2 \mathbb{P}\left(\max_{1 \le k \le n} \left|\sum_{i=1}^{k} X_i\right| > \varepsilon\right)}{1 - \mathbb{P}\left(\max_{1 \le k \le n} \left|\sum_{i=1}^{k} X_i\right| > \varepsilon\right)}.$$

3 Sharp sufficient conditions for mean convergence for the maximal partial sums

In this section, we provide sharp sufficient conditions for mean convergence for the maximal partial sums from arrays of random variables satisfying Condition (M_2).

Theorem 3.1 *Let $1 \le p < 2$ and let $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ be a triangular array of integrable random variables such that for each $n \ge 1$, the collection $\{X_{n,i}, 1 \le i \le k_n\}$ satisfies Condition (M_2). Let $\{b_n, n \ge 1\}$ be an increasing sequence of positive constants with $\lim_{n\to\infty} b_n = \infty$. If there exist a constant $C_1 > 0$ and $\delta \in (0, 2-p)$ such that*

$$\sup_{n \ge 1} \frac{1}{b_n^p} \sum_{i=1}^{k_n} \mathbb{E}\left(|X_{n,i}|^p \mathbf{1}\left(|X_{n,i}| > b_n\varepsilon\right)\right) \le \frac{C_1}{\varepsilon^\delta} \text{ for all } 0 < \varepsilon < 1 \tag{3.1}$$

and

$$\lim_{n\to\infty} \frac{1}{b_n^p} \sum_{i=1}^{k_n} \mathbb{E}\left(|X_{n,i}|^p \mathbf{1}\left(|X_{n,i}| > b_n\varepsilon\right)\right) = 0 \text{ for all } \varepsilon > 0, \tag{3.2}$$

then

$$\frac{1}{b_n} \max_{1 \le k \le k_n} \left|\sum_{i=1}^{k} \left(X_{n,i} - \mathbb{E}X_{n,i}\right)\right| \xrightarrow{\mathcal{L}_p} 0 \ \text{ as } n \to \infty. \tag{3.3}$$

Conversely, if $\mathbb{E}X_{n,i} \equiv 0$ and (3.3) holds, then so does (3.2).

Before presenting the proof of Theorem 3.1, we would like to provide some comments on conditions (3.1) and (3.2). While conditions (3.1) and (3.2) appear to be similar, they are independent in the sense that neither implies the other. The first example shows that (3.1) holds while (3.2) does not.

Example 3.2 Let $k_n \equiv n$, $1 \le p < 2$, and $b_n \equiv n^{1/p}$. Let $\{X_n, n \ge 1\}$ be a sequence of symmetric random variables such that

$$\mathbb{P}(X_n = 0) = 1 - \frac{1}{\log n} \ \text{ and } \ \mathbb{P}(X_n = \pm n^{1/p} \log^{1/p} n) = \frac{1}{2\log n}, n \ge 1.$$

Let $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ be a triangular array of random variables such that for all $n \ge 1$,

$$X_{n,i} = \begin{cases} 0 & \text{if } 1 \le i < n, \\ X_n & \text{if } i = n. \end{cases}$$

Then for all $n \geq 1$, we have

$$\begin{aligned}\frac{1}{b_n^p}\sum_{i=1}^{k_n}\mathbb{E}\left(|X_{n,i}|^p\mathbf{1}\left(|X_{n,i}|>b_n\varepsilon\right)\right)&=\frac{1}{n}\mathbb{E}\left(|X_n|^p\mathbf{1}\left(|X_n|>n^{1/p}\varepsilon\right)\right)\\&=\frac{1}{n}\mathbb{E}\left(|X_n|^p\right)\\&=1\leq\frac{1}{\varepsilon^\delta}\text{ for all }\delta>0\text{ and }\varepsilon\in(0,1).\end{aligned}$$

This implies that (3.1) holds but (3.2) fails.

In the next example, (3.1) fails but (3.2) holds. Example 3.3 also shows that condition (3.1) is sharp in the sense that Theorem 3.1 may fail if (3.1) is slightly weakened to

$$\sup_{n\geq 1}\frac{1}{b_n^p}\sum_{i=1}^{k_n}\mathbb{E}\left(|X_{n,i}|^p\mathbf{1}\left(|X_{n,i}|>b_n\varepsilon\right)\right)\leq\frac{C_2}{\varepsilon^{2-p}}\text{ for all }0<\varepsilon<1, \tag{3.4}$$

where C_2 is a positive constant.

Example 3.3 Let $k_n \equiv n$, $p = 1$, and $b_n = n^{1/p} = n$, $n \geq 1$. Let $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ be a array of independent symmetric random variables such that

$$\mathbb{P}(X_{n,i}=0)=1-\frac{1}{i}\text{ and }\mathbb{P}(X_{n,i}=\pm n/\log^{1/2}n)=\frac{1}{2i},\ 1\leq i\leq n, n\geq 1.$$

Let $\varepsilon \in (0, 1)$ be arbitrary. Then for all n satisfying $\log n \geq 1/\varepsilon^2$, we have

$$\begin{aligned}\frac{1}{b_n^p}\sum_{i=1}^{k_n}\mathbb{E}\left(|X_{n,i}|^p\mathbf{1}\left(|X_{n,i}|>b_n\varepsilon\right)\right)&=\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(|X_{n,i}|\mathbf{1}\left(|X_{n,i}|>n\varepsilon\right)\right)\\&=0\end{aligned} \tag{3.5}$$

implying that (3.2) holds. For all n satisfying $\log n < 1/\varepsilon^2$, we have

$$\begin{aligned}\frac{1}{b_n^p}\sum_{i=1}^{k_n}\mathbb{E}\left(|X_{n,i}|^p\mathbf{1}\left(|X_{n,i}|>b_n\varepsilon\right)\right)&=\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(|X_{n,i}|\mathbf{1}\left(|X_{n,i}|>n\varepsilon\right)\right)\\&=\frac{1}{\log^{1/2}n}\sum_{i=1}^{n}\frac{1}{i}\\&\leq 10\log^{1/2}n\leq\frac{10}{\varepsilon},\end{aligned} \tag{3.6}$$

which, together with (3.5), implies that (3.4) holds with $C_2 = 10$.

Now, for any sufficiently small $\varepsilon > 0$, and for all n satisfying $1/\varepsilon^2 \leq 4\log n < 4/\varepsilon^2$, we have

$$\begin{aligned}\frac{1}{b_n^p}\sum_{i=1}^{k_n}\mathbb{E}\left(|X_{n,i}|^p\mathbf{1}\left(|X_{n,i}|>b_n\varepsilon\right)\right)&=\frac{1}{\log^{1/2}n}\sum_{i=1}^{n}\frac{1}{i}\\&\geq\log^{1/2}n\geq\frac{1}{2\varepsilon}\end{aligned}$$

showing that (3.1) fails for every universal constants C_1 and $\delta \in (0, 1)$.

Finally, we will show that (3.3) fails. It is clear that for all $n \geq 1$,

$$\max_{1 \leq i \leq n} \frac{|X_{n,i}|}{n} \leq 1 \text{ a.s.}$$

Applying Lemma 2.2 with $\varepsilon = 1/4$, we have for all $n \geq 1$,

$$\frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E}X_{n,i}^2 \leq \frac{1}{16} + \frac{25}{16} \times \frac{\mathbb{P}\left(\max_{1 \leq k \leq n} \left|\sum_{i=1}^{k} X_{n,i}\right| > n/4\right)}{1 - \mathbb{P}\left(\max_{1 \leq k \leq n} \left|\sum_{i=1}^{k} X_i\right| > n/4\right)}. \tag{3.7}$$

If (3.3) holds, then there exists n_0 such that $\mathbb{P}\left(\max_{1 \leq k \leq n} \left|\sum_{i=1}^{k} X_{n,i}\right| > n/4\right) \leq 1/4$ for all $n \geq n_0$. Therefore, we have from (3.7) that

$$\frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E}X_{n,i}^2 \leq \frac{1}{16} + \frac{25}{48} < 1 \text{ for all } n \geq n_0. \tag{3.8}$$

However, we have

$$\frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E}X_{n,i}^2 = \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} \geq 1, \; n \geq 1$$

contradicting (3.8). Therefore, (3.3) must fail.

The rest of this section is devoted to proving Theorem 3.1.

Proof of Theorem 3.1. Firstly, we prove the sufficiency. Assume that (3.1) and (3.2) hold. For $n \geq 1$, $1 \leq i \leq k_n$ and $t > 0$, set

$$Y_{n,i,t} = -b_n t^{1/p} \mathbf{1}(X_{n,i} < -b_n t^{1/p}) + X_{n,i} \mathbf{1}(|X_{n,i}| \leq b_n t^{1/p}) + b_n t^{1/p} \mathbf{1}(X_{n,i} > b_n t^{1/p}).$$

Then it follows from (3.2) that

$$\begin{aligned} \sup_{t \geq 1} \frac{1}{b_n t^{1/p}} \sum_{i=1}^{k_n} \mathbb{E}|X_{n,i} - Y_{n,i,t}| &\leq \frac{1}{b_n} \sum_{i=1}^{k_n} \mathbb{E}\left(|X_{n,i}| \mathbf{1}\left(|X_{n,i}| > b_n\right)\right) \\ &\leq \frac{1}{b_n^p} \sum_{i=1}^{k_n} \mathbb{E}\left(|X_{n,i}|^p \mathbf{1}\left(|X_{n,i}| > b_n\right)\right) \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.9}$$

Let $\varepsilon_0 > 0$ be arbitrary. Then for all large n and for all $t \geq 1$, we have

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{b_n}\max_{1\leq k\leq k_n}\left|\sum_{i=1}^{k}(X_{n,i}-\mathbb{E}X_{n,i})\right| > \varepsilon_0\right)\\
&\leq \sum_{i=1}^{k_n}\mathbb{P}\left(|X_{n,i}| > b_n t^{1/p}\right)\\
&\quad + \mathbb{P}\left(\max_{1\leq k\leq k_n}\left|\sum_{i=1}^{k}(Y_{n,i,t}-\mathbb{E}X_{n,i})\right| > b_n\varepsilon_0\right)\\
&\leq o(1) + \mathbb{P}\left(\max_{1\leq k\leq k_n}\left|\sum_{i=1}^{k}(Y_{n,i,t}-\mathbb{E}Y_{n,i,t})\right| > b_n\varepsilon_0/2\right)\\
&\leq o(1) + \frac{4}{b_n^2\varepsilon_0^2}\mathbb{E}\left(\max_{1\leq k\leq k_n}\left|\sum_{i=1}^{k}(Y_{n,i,t}-\mathbb{E}Y_{n,i,t})\right|\right)^2\\
&\leq o(1) + \frac{4C_0}{b_n^2\varepsilon_0^2}\sum_{i=1}^{k_n}\mathbb{E}Y_{n,i,t}^2,
\end{aligned} \tag{3.10}$$

where we have applied (3.2) and (3.9) in the second inequality, and (2.1) in the last inequality. We will now estimate the last term in (3.10). Let $0 < \varepsilon < 1/2$ be arbitrary. Then

$$\begin{aligned}
\frac{1}{b_n^2}\sum_{i=1}^{k_n}\mathbb{E}Y_{n,i,1}^2 &\leq \sum_{i=1}^{k_n}\frac{1}{b_n^2}\int_0^{\varepsilon^2 b_n^2}\mathbb{P}\left(|X_{n,i}| > u^{1/2}\right)\mathrm{d}u\\
&\quad + \sum_{i=1}^{k_n}\frac{1}{b_n^2}\int_{\varepsilon^2 b_n^2}^{b_n^2}\mathbb{P}\left(|X_{n,i}| > u^{1/2}\right)\mathrm{d}u\\
&\leq \sum_{i=1}^{k_n}\int_0^{\varepsilon^2}\mathbb{P}\left(|X_{n,i}| > x^{1/2}b_n\right)\mathrm{d}x\\
&\quad + \sum_{i=1}^{k_n}\frac{1}{b_n^2}\int_{\varepsilon^2 b_n^2}^{b_n^2}\mathbb{P}\left(|X_{n,i}| > \varepsilon b_n\right)\mathrm{d}u\\
&\leq \int_0^{\varepsilon^2} b_n^{-p}x^{-p/2}\sum_{i=1}^{k_n}\mathbb{E}\left(|X_{n,i}|^p\mathbf{1}\left(|X_{n,i}| > x^{1/2}b_n\right)\right)\mathrm{d}x\\
&\quad + \sum_{i=1}^{k_n}\mathbb{P}\left(|X_{n,i}| > \varepsilon b_n\right)\\
&\leq C_1\int_0^{\varepsilon^2}\frac{\mathrm{d}x}{x^{(p+\delta)/2}} + \sum_{i=1}^{k_n}\mathbb{P}\left(|X_{n,i}| > \varepsilon b_n\right)\\
&= \frac{2C_1\varepsilon^{2-p-\delta}}{(2-p-\delta)} + \sum_{i=1}^{k_n}\mathbb{P}\left(|X_{n,i}| > \varepsilon b_n\right).
\end{aligned} \tag{3.11}$$

It is clear that (3.2) implies

$$\lim_{n\to\infty}\sum_{i=1}^{k_n}\mathbb{P}\left(|X_{n,i}|>\varepsilon b_n\right)=0 \text{ for all } \varepsilon>0. \tag{3.12}$$

Since $0<\varepsilon<1/2$ is arbitrary and $2-p-\delta>0$, it follows from (3.10), (3.11), (3.12) that

$$\mathbb{P}\left(\frac{1}{b_n}\max_{1\le k\le k_n}\left|\sum_{i=1}^{k}(X_{n,i}-\mathbb{E}X_{n,i})\right|>\varepsilon_0\right)\to 0 \text{ as } n\to\infty \text{ for all } \varepsilon_0>0. \tag{3.13}$$

Let

$$I(n)=\int_0^1\mathbb{P}\left(\frac{1}{b_n}\max_{1\le k\le k_n}\left|\sum_{i=1}^{k}(X_{n,i}-\mathbb{E}X_{n,i})\right|>t^{1/p}\right)\mathrm{d}\,t,\ n\ge 1.$$

By (3.13) and the Lebesgue dominated convergence theorem, we obtain

$$I(n)\to 0 \text{ as } n\to\infty. \tag{3.14}$$

For all n sufficiently large, we have

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{b_n}\max_{1\le k\le k_n}\left|\sum_{i=1}^{k}(X_{n,i}-\mathbb{E}X_{n,i})\right|\right)^p\\
&=I(n)+\int_1^\infty\mathbb{P}\left(\frac{1}{b_n}\max_{1\le k\le k_n}\left|\sum_{i=1}^{k}(X_{n,i}-\mathbb{E}X_{n,i})\right|>t^{1/p}\right)\mathrm{d}\,t\\
&\le I(n)+\int_1^\infty\sum_{i=1}^{k_n}\mathbb{P}\left(|X_{n,i}|>b_nt^{1/p}\right)\mathrm{d}\,t\\
&\quad+\int_1^\infty\mathbb{P}\left(\max_{1\le k\le k_n}\left|\sum_{i=1}^{k}(Y_{n,i,t}-\mathbb{E}Y_{n,i,t})\right|>b_nt^{1/p}/2\right)\mathrm{d}\,t \qquad (3.15)\\
&\le I(n)+\frac{1}{b_n^p}\sum_{i=1}^{k_n}\mathbb{E}\left(|X_{n,i}|^p\mathbf{1}\left(|X_{n,i}|>b_n\right)\right)\\
&\quad+\int_1^\infty\mathbb{P}\left(\max_{1\le k\le k_n}\left|\sum_{i=1}^{k}(Y_{n,i,t}-\mathbb{E}Y_{n,i,t})\right|>b_nt^{1/p}/2\right)\mathrm{d}\,t\\
&=o(1)+\int_1^\infty\mathbb{P}\left(\max_{1\le k\le k_n}\left|\sum_{i=1}^{k}(Y_{n,i,t}-\mathbb{E}Y_{n,i,t})\right|>b_nt^{1/p}/2\right)\mathrm{d}\,t,
\end{aligned}$$

where we have used the same estimate as the first two inequalities of (3.10) in the first inequality, and (3.2) and (3.14) in the last step. To complete the proof of the sufficiency part, it remains to prove that the last term in (3.15) also converges to 0. By using the same estimate as the last two inequalities of (3.10) and Tonelli's theorem, we have

$$\begin{aligned}
&\int_1^\infty \mathbb{P}\left(\max_{1\le k\le k_n}\left|\sum_{i=1}^{k}(Y_{n,i,t}-\mathbb{E}Y_{n,i,t})\right| > b_n t^{1/p}/2\right)\mathrm{d}t \le 4C_0\sum_{i=1}^{k_n}\int_1^\infty \frac{\mathbb{E}Y_{n,i,t}^2}{b_n^2 t^{2/p}}\mathrm{d}t\\
&= 4C_0\sum_{i=1}^{k_n}\int_1^\infty \frac{1}{t^{2/p}}\left(\int_0^{t^{2/p}}\mathbb{P}\left(|b_n^{-1}X_{n,i}|>u^{1/2}\right)\mathrm{d}u\right)\mathrm{d}t\\
&= 4C_0\sum_{i=1}^{k_n}\int_1^\infty \frac{1}{t^{2/p}}\left(\int_0^{1}\mathbb{P}\left(|X_{n,i}|>b_n u^{1/2}\right)\mathrm{d}u+\int_1^{t^{2/p}}\mathbb{P}\left(|X_{n,i}|>b_n u^{1/2}\right)\mathrm{d}u\right)\mathrm{d}t\\
&= \frac{4pC_0}{2-p}\sum_{i=1}^{k_n}\left(\int_0^{1}\mathbb{P}\left(|X_{n,i}|>b_n u^{1/2}\right)\mathrm{d}u+\int_1^{\infty}u^{p/2-1}\mathbb{P}\left(|X_{n,i}|>b_n u^{1/2}\right)\mathrm{d}u\right)\\
&= \frac{4pC_0}{2-p}\sum_{i=1}^{k_n}\left(\int_0^{1}\mathbb{P}\left(|X_{n,i}|>b_n u^{1/2}\right)\mathrm{d}u+\int_1^{\infty}\mathbb{P}\left(|X_{n,i}|>b_n x^{1/p}\right)\mathrm{d}x\right)\\
&\le \frac{4pC_0}{2-p}\sum_{i=1}^{k_n}\left(\int_0^{1}\mathbb{P}\left(|X_{n,i}|>b_n u^{1/2}\right)\mathrm{d}u+\frac{1}{b_n^p}\mathbb{E}\left(|X_{n,i}|^p\mathbf{1}\left(|X_{n,i}|>b_n\right)\right)\right).
\end{aligned} \tag{3.16}$$

From (3.1), we have for all $u\in(0,1)$ that

$$\begin{aligned}
\sup_{n\ge1}\sum_{i=1}^{k_n}\mathbb{P}\left(|X_{n,i}|>b_n u^{1/2}\right) &\le \sup_{n\ge1}\frac{1}{b_n^p u^{p/2}}\sum_{i=1}^{k_n}\mathbb{E}\left(|X_{n,i}|^p\mathbf{1}\left(|X_{n,i}|>b_n u^{1/2}\right)\right)\\
&\le \frac{C_1}{u^{(p+\delta)/2}}.
\end{aligned} \tag{3.17}$$

Since $p+\delta<2$, the function $f(u)=C_1/u^{(p+\delta)/2}$ is integrable on $(0,1)$. It thus follows from (3.12), (3.17), and the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty}\int_0^1\sum_{i=1}^{k_n}\mathbb{P}\left(|X_{n,i}|>b_n u^{1/2}\right)\mathrm{d}u=0. \tag{3.18}$$

Combining (3.16), (3.2) and (3.18) yields

$$\lim_{n\to\infty}\int_1^\infty \mathbb{P}\left(\max_{1\le k\le k_n}\left|\sum_{i=1}^{k}(Y_{n,i,t}-\mathbb{E}Y_{n,i,t})\right| > b_n t^{1/p}/2\right)\mathrm{d}t=0. \tag{3.19}$$

Combining (3.15) and (3.19) completes the proof of the sufficiency part of the theorem.

Finally, we prove the necessity. Assume that $\mathbb{E}X_{n,i}\equiv 0$ and (3.3) holds. Let $\varepsilon\in(0,1)$ be arbitrary. Since

$$\max_{1\le i\le k_n}|X_{n,i}|\le 2\max_{1\le k\le k_n}\left|\sum_{i=1}^{k}X_{n,i}\right|,\ n\ge1,$$

we obtain from (3.3) that

$$\frac{1}{b_n}\max_{1\le i\le k_n}\left|X_{n,i}\right|\overset{\mathcal{L}_p}{\to}0\ \text{ as } n\to\infty. \tag{3.20}$$

By (3.20) and the second part of Lemma 2.1, there exists a positive integer n_0 depending only on ε such that

$$\sum_{i=1}^{k_n} \mathbb{P}\left(|X_{n,i}| > b_n x\right) \le 4C_2 \mathbb{P}\left(\max_{1\le i\le k_n} |X_{n,i}| > b_n x\right) \text{ for all } x \ge \varepsilon, n \ge n_0,$$

where C_2 is a constant depending only on C_0. Therefore, for all $n \ge n_0$, we have

$$\begin{aligned}
&\frac{1}{b_n^p}\sum_{i=1}^{k_n} \mathbb{E}\left(|X_{n,i}|^p \mathbf{1}\left(|X_{n,i}| > b_n\varepsilon\right)\right) \\
&= \sum_{i=1}^{k_n}\left(\int_{\varepsilon^p}^{\infty} \mathbb{P}\left(|X_{n,i}| > b_n x^{1/p}\right) \mathrm{d}\,x + \varepsilon^p \mathbb{P}\left(|X_{n,i}| > b_n\varepsilon\right)\right) \\
&\le 4C_2\left(\int_{\varepsilon^p}^{\infty} \mathbb{P}\left(\max_{1\le i\le k_n} |X_{n,i}| > b_n x^{1/p}\right) \mathrm{d}\,x + \varepsilon^p \mathbb{P}\left(\max_{1\le i\le k_n} |X_{n,i}| > b_n\varepsilon\right)\right) \\
&= \frac{4C_2}{b_n^p}\mathbb{E}\left(\max_{1\le i\le k_n} |X_{n,i}|^p \mathbf{1}\left(\max_{1\le i\le k_n} |X_{n,i}| > b_n\varepsilon\right)\right) \\
&\le \frac{4C_2}{b_n^p}\mathbb{E}\left(\max_{1\le i\le k_n} |X_{n,i}|\right)^p. \qquad (3.21)
\end{aligned}$$

Combining (3.20) and (3.21) yields (3.2). The proof of the theorem is completed. □

Remark 3.4 Since Lemma 2.1 holds under the assumption that the collection $\{X_{n,i}, 1 \le i \le k_n\}$ satisfies Condition (H_2) for all $n \ge 1$, the same holds for the necessary part of Theorem 3.1.

4 Mean convergence for the maximal partial sums under regularly varying moment conditions

In this section, we will apply the sufficiency part of Theorem 3.1 to give a positive answer to Question 1. The main result of this section is the following theorem.

Theorem 4.1 *Let $1 \le p < 2$ and let $L(\cdot)$ be an increasing slowly varying function defined on $[0, \infty)$ such that $L(x) \ge 1$ for $x \ge 0$. Let $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ be a triangular array of integrable random variables such that for all $n \ge 1$, the collection $\{X_{n,i}, 1 \le i \le n\}$ satisfies Condition (M_2). If the array $\{|X_{n,i}|^p L(|X_{n,i}|^p), 1 \le i \le n, n \ge 1\}$ is uniformly integrable in the Cesàro sense, then*

$$\frac{1}{n^{1/p}\tilde{L}^{1/p}(n)} \max_{1\le k\le n}\left|\sum_{i=1}^{k}(X_{n,i} - \mathbb{E}X_{n,i})\right| \xrightarrow{\mathcal{L}_p} 0 \text{ as } n \to \infty. \qquad (4.1)$$

Proof Let $b_n \equiv n^{1/p}\tilde{L}^{1/p}(n)$ and $k_n \equiv n$. By the sufficiency part of Theorem 3.1, we only have to show that (3.1) and (3.2) are satisfied. To do this, we will use the same argument as in the proof of Claim 1 in [19]. Firstly, we will verify (3.1). By the second half of (1.1), we have $\lim_{n\to\infty} \tilde{L}(n)L(n\tilde{L}(n)) = 1$. It implies that

$$\sup_{n\ge 1} \frac{1}{\tilde{L}(n)L(n\tilde{L}(n))} := c_1 < \infty. \qquad (4.2)$$

Since $L(x) \geq 1$ for all $x \geq 0$ and $\{|X_{n,i}|^p L(|X_{n,i}|^p), 1 \leq i \leq n, n \geq 1\}$ is uniformly integrable in the Cesàro sense,

$$\sup_{n\geq 1} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left(|X_{n,i}|^p\right) \leq \sup_{n\geq 1} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left(|X_{n,i}|^p L(|X_{n,i}|^p)\right) := c_2 < \infty. \tag{4.3}$$

Let $\beta \in (0, 2/p - 1)$. By the Potter bound (see, e.g., Theorem B.1.9 (5) in [10]), there exists $a_0 > 0$ such that for all $\varepsilon \in (0, 1)$ and for all $x \geq a_0$, we have

$$L(x/\varepsilon) \leq \frac{4L(x)}{\varepsilon^{\beta}}. \tag{4.4}$$

By using the monotonicity of the function $L(x)$ and (4.2)–(4.4), we have for all $\varepsilon \in (0, 1)$ and $n \geq 1$,

$$\begin{aligned}
&\frac{1}{b_n^p} \sum_{i=1}^{n} \mathbb{E}\left(|X_{n,i}|^p \mathbf{1}\left(|X_{n,i}| > b_n\varepsilon\right)\right)\\
&\leq \frac{1}{b_n^p L(b_n^p)} \sum_{i=1}^{n} \mathbb{E}\left(|X_{n,i}|^p L(|X_{n,i}|^p/\varepsilon^p) \mathbf{1}\left(|X_{n,i}| > b_n\varepsilon\right)\right)\\
&\leq \frac{1}{n\tilde{L}(n)L(n\tilde{L}(n))} \sum_{i=1}^{n} \mathbb{E}\left(|X_{n,i}|^p L(a_0/\varepsilon^p)\right)\\
&\quad + \frac{1}{n\tilde{L}(n)L(n\tilde{L}(n))} \sum_{i=1}^{n} \mathbb{E}\left(|X_{n,i}|^p L(|X_{n,i}|^p/\varepsilon^p) \mathbf{1}\left(|X_{n,i}|^p > a_0\right)\right)\\
&\leq \frac{4c_1}{n} \sum_{i=1}^{n} \left(\frac{L(a_0)\mathbb{E}\left(|X_{n,i}|^p\right)}{\varepsilon^{p\beta}} + \frac{\mathbb{E}\left(|X_{n,i}|^p L(|X_{n,i}|^p)\right)}{\varepsilon^{p\beta}} \right)\\
&\leq \frac{4c_1c_2(L(a_0)+1)}{\varepsilon^{p\beta}}.
\end{aligned} \tag{4.5}$$

By choosing $C_1 = 4c_1c_2(L(a_0) + 1)$ and $\delta = p\beta \in (0, 2 - p)$, we obtain (3.1).

Next, we will verify (3.2). Let $\varepsilon \in (0, 1)$ be arbitrary. Proceeding in a similar manner as in (4.5), we have for all $n \geq 1$,

$$\begin{aligned}
\frac{1}{b_n^p} \sum_{i=1}^{n} \mathbb{E}\left(|X_{n,i}|^p \mathbf{1}\left(|X_{n,i}| > b_n\varepsilon\right)\right) \leq \frac{4c_1}{n\varepsilon^{p\beta}} \sum_{i=1}^{n} \Big(& L(a_0)\mathbb{E}\left(|X_{n,i}|^p \mathbf{1}(|X_{n,i}| > \varepsilon b_n)\right)\\
&+\mathbb{E}\left(|X_{n,i}|^p L(|X_{n,i}|^p)\mathbf{1}(|X_{n,i}| > \varepsilon b_n)\right)\Big).
\end{aligned} \tag{4.6}$$

Since $L(x) \geq 1$ and $\{|X_{n,i}|^p L(|X_{n,i}|^p), 1 \leq i \leq n, n \geq 1\}$ is uniformly integrable in the Cesàro sense, we have

$$\lim_{n\to\infty} \sup_{m\geq 1} \frac{1}{m} \sum_{i=1}^{m} \mathbb{E}\left(|X_{m,i}|^p L(|X_{m,i}|^p)\mathbf{1}(|X_{m,i}| > \varepsilon b_n)\right) = 0, \tag{4.7}$$

and

$$\lim_{n\to\infty} \sup_{m\geq 1} \frac{1}{m} \sum_{i=1}^{m} \mathbb{E}\left(|X_{m,i}|^p \mathbf{1}(|X_{m,i}| > \varepsilon b_n)\right) = 0. \tag{4.8}$$

Combining (4.6), (4.7) and (4.8) yields (3.2). The proof of the theorem is completed. □

Concerning Theorem 4.1, a Reviewer kindly raised a question as to whether or not the assumptions that $L(\cdot)$ is increasing and $L(x) \ge 1$ for $x \ge 0$ can be removed for the case $1 < p < 2$. This leads to the following example. It shows that Theorem 4.1 may fails if $L(x) \downarrow 0$ as $x \to \infty$.

Example 4.2 Let $1 < p < 2$ and let $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ be an array of integrable, symmetric, independent and identically distributed random variables such that $\mathbb{E}(|X_{1,1}|^p \log^{-1}|X_{1,1}|) < \infty$ and $\mathbb{E}|X_{1,1}|^p = \infty$. Let $L(x) = \log^{-1} x$, $x \ge 0$. Then all assumptions of Theorem 4.1 are satisfied, except that $L(x)$ is not increasing, and $\lim_{x\to\infty} L(x) = 0$. Since $\mathbb{E}X_{n,i} \equiv 0$ and for all $n \ge 1$,

$$\max_{1\le i\le n} |X_{n,i}| \le 2 \max_{1\le k\le n} \left| \sum_{i=1}^{k} X_{n,i} \right|,$$

we have

$$\begin{aligned}\mathbb{E}\left(\frac{2}{n^{1/p}\tilde{L}^{1/p}(n)} \max_{1\le k\le n} \left|\sum_{i=1}^{k}(X_{n,i} - \mathbb{E}X_{n,i})\right|\right)^p &\ge \frac{1}{n\log n}\mathbb{E}\left(\max_{1\le i\le n}|X_{n,i}|^p\right)\\ &\ge \frac{1}{n\log n}\mathbb{E}|X_{1,1}|^p = \infty\end{aligned}$$

so that (4.1) fails.

Remark 4.3 We note that in the context of the Marcinkiewicz–Zygmund strong and weak law of large numbers, the assumptions that $L(\cdot)$ is increasing and $L(x) \ge 1$ for $x \ge 0$ are required only for the case $p = 1$. We refer to Theorem 3.1 in [2], Theorem 1 in [16], and Corollary 4.10 in [18] for details.

The following corollary is a special case of Theorem 4.1. To our best knowledge, this result is also new even in the independence case. Almost sure convergence for sequences of negatively associated random variables under a similar moment condition was studied by Miao et al. [12].

Corollary 4.4 *Let $\alpha \ge 0$, $1 \le p < 2$ and let $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ be a triangular array of integrable random variables such that for all $n \ge 1$, the collection $\{X_{n,i}, 1 \le i \le n\}$ satisfies Condition (M_2). If $\{|X_{n,i}|^p(\log|X_{n,i}|)^\alpha, 1 \le i \le n, n \ge 1\}$ is uniformly integrable in the Cesàro sense, then*

$$\frac{(\log n)^{\alpha/p}}{n^{1/p}} \max_{1\le k\le n} \left|\sum_{i=1}^{k}\left(X_{n,i} - \mathbb{E}X_{n,i}\right)\right| \xrightarrow{\mathcal{L}_p} 0 \text{ as } n \to \infty. \tag{4.9}$$

Proof By choosing $L(x) \equiv \log^\alpha x$, we have $\tilde{L}(x) \equiv \log^{-\alpha} x$. The proof thus follows by applying Theorem 4.1. □

The following example shows that in Corollary 4.4, the assumption that $\{|X_{n,i}|^p(\log|X_{n,i}|)^\alpha, 1 \le i \le n, n \ge 1\}$ is uniformly integrable in the Cesàro sense cannot be replaced with

$$\sup_{1\le i\le n, n\ge 1} \mathbb{E}(|X_{n,i}|^p(\log|X_{n,i}|)^\alpha) < \infty. \tag{4.10}$$

Example 4.5 Let $1 \leq p < 2$, $\alpha \geq 0$, and let $\{X_n, n \geq 1\}$ be a sequence of independent symmetric random variables with

$$\mathbb{P}(X_n = 0) = 1 - \frac{1}{n},\ \mathbb{P}(X_n = n^{1/p}\log^{-\alpha/p} n) = \mathbb{P}(X_n = -n^{1/p}\log^{-\alpha/p} n) = \frac{1}{2n},\ n \geq 1.$$

Then it is clear that

$$\begin{aligned}\sup_{n\geq 1}\mathbb{E}(|X_n|^p \log^\alpha |X_n|) &= \sup_{n\geq 1}\left(n\log^{-\alpha} n\right)\log^\alpha\left(n^{1/p}\log^{-\alpha/p} n\right) \times \frac{1}{n}\\ &= \sup_{n\geq 1}\left(\frac{\log\left(n^{1/p}\log^{-\alpha/p} n\right)}{\log n}\right)^\alpha \\ &\leq \sup_{n\geq 1}\left(\frac{\log\left(n^{1/p}\right)}{\log n}\right)^\alpha = \left(\frac{1}{p}\right)^\alpha < \infty.\end{aligned} \tag{4.11}$$

For all n satisfying $n^{1/p}\log^{-\alpha/p} n \geq n^{1/(2p)}$, we have

$$\begin{aligned}&\mathbb{E}(|X_n|^p(\log^\alpha |X_n|)\mathbf{1}(|X_n|^p(\log^\alpha |X_n|) > a))\\ &= \left(n\log^{-\alpha} n\right)\log^\alpha\left(n^{1/p}\log^{-\alpha/p} n\right) \times \frac{1}{n}\\ &= \left(\frac{\log\left(n^{1/p}\log^{-\alpha/p} n\right)}{\log n}\right)^\alpha\\ &\geq \left(\frac{\log\left(n^{1/(2p)}\right)}{\log n}\right)^\alpha = \left(\frac{1}{2p}\right)^\alpha \quad \text{for all } a > 0.\end{aligned} \tag{4.12}$$

Let $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of random variables with $X_{n,i} = X_i$ for all $1 \leq i \leq n, n \geq 1$. Then by (4.11) and (4.12), we see that (4.10) is satisfied but $\{|X_{n,i}|^p(\log|X_{n,i}|)^\alpha, 1 \leq i \leq n, n \geq 1\}$ is not uniformly integrable in the Cesàro sense. Let $b_n \equiv n^{1/p}\log^{-\alpha/p} n$. Then for $0 < \varepsilon < 1/4$ and for $n \geq 2$, we have

$$\begin{aligned}\sum_{i=1}^n \mathbb{P}(|X_i| > \varepsilon b_n) &\geq \sum_{i=\lfloor n/2\rfloor}^n \mathbb{P}(|X_i| > \varepsilon b_n)\\ &\geq \sum_{i=\lfloor n/2\rfloor}^n \frac{1}{n} \geq \frac{1}{2},\end{aligned} \tag{4.13}$$

so that (3.2) fails. Applying the necessity part of Theorem 3.1 with $k_n \equiv n$, we have

$$\frac{\max_{1\leq k\leq n}\left|\sum_{i=1}^k X_i\right|}{b_n} \overset{\mathcal{L}_p}{\nrightarrow} 0,$$

i.e., (4.9) fails.

5 Mean convergence for the partial sums from arrays of rowwise and pairwise negatively dependent random variables

Mean convergence theorems for pairwise negatively dependent random variables were studied by various authors. Ordóñez Cabrera and Volodin [13, Theorem 1] studied $\mathcal{L}_1$ convergence theorem for weighted sums from arrays $\{X_{n,i}, u_n \le i \le v_n, n \ge 1\}$ of rowwise and pairwise negatively dependent random variables satisfying the so-called h-integrability concerning the array of weights, where $\{h(n), n \ge 1\}$ is an increasing sequence of positive constants. Here and hereafter, $\{u_n, n \ge 1\}$ and $\{v_n, n \ge 1\}$ denote two sequences of integer numbers such that $u_n < v_n$ for all $n \ge 1$ and $\lim(v_n - u_n) = \infty$. Wu and Guan [22] extended Theorem 1 of Ordóñez Cabrera and Volodin [13] to $\mathcal{L}_p$ convergence, $1 \le p < 2$. To this end, Sung [15, Theorem 2.1] extended these results by proving the following theorem.

Theorem 5.1 (Sung [15], Theorem 2.1) *Let $1 \le p < 2$ and let $\{X_{n,i}, u_n \le i \le v_n, n \ge 1\}$ be a triangular array of rowwise and pairwise negatively dependent random variables. Let $\{b_n, n \ge 1\}$ be a sequence of increasing to infinity of positive constants. Suppose that*

$$\sup_{n\ge1} \frac{1}{b_n^p} \sum_{i=u_n}^{v_n} \mathbb{E}|X_{n,i}|^p < \infty \tag{5.1}$$

and

$$\lim_{n\to\infty} \frac{1}{b_n^p} \sum_{i=u_n}^{v_n} \mathbb{E}\left(|X_{n,i}|^p \mathbf{1}\left(|X_{n,i}| > b_n\varepsilon\right)\right) = 0 \text{ for all } \varepsilon > 0, \tag{5.2}$$

then

$$\frac{1}{b_n} \sum_{i=u_n}^{v_n} \left(X_{n,i} - \mathbb{E}X_{n,i}\right) \xrightarrow{\mathcal{L}_p} 0 \text{ as } n \to \infty. \tag{5.3}$$

In Theorem 2.1 of Sung [15], the author stated the result for weighted sums $\sum_{i=u_n}^{v_n} a_{n,i}\left(Y_{n,i} - \mathbb{E}Y_{n,i}\right)$, but if we let $X_{n,i} \equiv b_n a_{n,i} Y_{n,i}$, then Theorem 5.1 coincides with his result. Since pairwise negative dependence enjoys Condition (H_2), we obtain following theorem by using the same steps in the proof of the sufficiency of Theorem 3.1 with only minor changes.

Theorem 5.2 *Theorem 5.1 still holds if (5.1) is weakened to*

$$\sup_{n\ge1} \frac{1}{b_n^p} \sum_{i=u_n}^{v_n} \mathbb{E}\left(|X_{n,i}|^p \mathbf{1}\left(|X_{n,i}| > b_n\varepsilon\right)\right) \le \frac{C_1}{\varepsilon^\delta} \text{ for all } 0 < \varepsilon < 1, \tag{5.4}$$

where $C_1 > 0$ and $\delta \in (0, 2-p)$ are some constants which do not depend on ε.

The following example which is inspired by Example 4.8 in Thành [18] shows that (5.4) is strictly weaker than (5.1).

Example 5.3 Let $1 \le p < 2$, $0 < \alpha < 1$, $b_n \equiv n^{1/p}\log^{-\alpha/p} n$, $u_n \equiv 1$, $v_n \equiv n$ and let $\{X_n, n \ge 1\}$ be a sequence of independent random variables such that

$$\mathbb{P}(X_n = -1) = \mathbb{P}(X_n = 1) = 1/2 \text{ for } n = 1 \text{ or } n \ne 2^m,\ m \ge 1,$$

and

$$\mathbb{P}\left(X_{2^m}=-2^{m/p}/m^{1/p}\right)=\mathbb{P}\left(X_{2^m}=2^{m/p}/m^{1/p}\right)=1/2 \text{ for } m\geq 1.$$

Let $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ of random variables such that $X_{n,i} = X_i$ for all $n \geq 1$, $1 \leq i \leq n$. For $n \geq 1$, we have from (4.35) of Thành [18] that

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(|X_{n,i}|^p\log(|X_{n,i}|)\right)\leq 1+\frac{2}{p}<\infty.$$

Therefore, $\{|X_{n,i}|^p \log^\alpha |X_{n,i}|, 1 \leq i \leq n, n \geq 1\}$ is uniformly integrable in the Cesàro sense by the de La Vallée Poussin criterion for the Cesàro uniform integrability (see Chandra and Goswami [6, p., 228–230]). Applying Theorem 4.1, we obtain the conclusion (4.1).

Finally, it is clear that for all $n \geq 1$, we have

$$\frac{1}{b_n^p}\sum_{i=u_n}^{v_n}\mathbb{E}|X_{n,i}|^p=\frac{\log^\alpha n}{n}\sum_{i=1}^{n}\mathbb{E}|X_i|^p\geq\log^\alpha n\to\infty.$$

Thus, (5.1) fails, and we cannot apply Theorem 2.1 of Sung [15] (Theorem 5.1) to derive even (5.3) which is clearly weaker than (4.1).

Remark 5.4 Since (5.4) is strictly weaker than (5.1), Theorem 5.2 improves Theorem 2.1 of Sung [15], which, in turn, improves Theorem 1 of Ordóñez Cabrera and Volodin [13] and Theorems 3.1-—3.3 of Wu and Guan [22].

Remark 5.5 Wu et al. [25] established mean convergence of the partial sums from triangular arrays of rowwise widely orthant dependent random variables. It would be interesting to see if one can use the method developed in this paper to give an improvement of Theorem 3.1 in [25] (by weakening condition (3) of Theorem 3.1 in [25]).

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Conflict of interest The author has no competing interests to declare.

References

1. Adler, A., Matuła, P.: On exact strong laws of large numbers under general dependence conditions. Probab. Math. Stat. **38**(1), 103–121 (2018)
2. Anh, V.T.N., Hien, N.T.T., Thành, L.V., Van, V.T.H.: The Marcinkiewicz-Zygmund-type strong law of large numbers with general normalizing sequences. J. Theory Probab. **34**(1), 331–348 (2021)
3. Bernou, I., Boukhari, F.: Limit theorems for dependent random variables with infinite means. Stat. Probab. Lett. **189**, 109563 (2022)
4. Bingham, N.H., Goldie, C.M., Teugels, J.L.: Regular Variation, vol. 27. Cambridge University Press, Cambridge (1989)

5. Chandra, T.K.: Uniform integrability in the Cesàro sense and the weak law of large numbers. Sankhyā Indian J. Stat. Ser. A **51**(3), 309–317 (1989)
6. Chandra, T.K., Goswami, A.: Cesàro uniform integrability and the strong law of large numbers. Sankhyā Indian J. Stat. Ser. A **54**(2), 215–231 (1992)
7. Chow, Y.S.: On the L_p-convergence for $n^{-1/p}S_n$, $0<p<2$. Ann. Math. Stat. **42**(1), 393–394 (1971)
8. Gut, A.: An extension of the Kolmogorov-Feller weak law of large numbers with an application to the St. Petersburg game. J. Theory Probab. **17**(3), 769–779 (2004)
9. Gut, A.: Probability: A Graduate Course, 2nd edn. Springer, New York (2013)
10. Haan, L., Ferreira, A.: Extreme Value Theory: An Introduction. Springer, New York (2006)
11. Jessen, H.A., Mikosch, T.: Regularly varying functions. Publications de L'institut Mathematique (Beograd) (N.S.) **80**(94), 171–192 (2006)
12. Miao, Y., Mu, J., Xu, J.: An analogue for Marcinkiewicz-Zygmund strong law of negatively associated random variables. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas **111**(3), 697–705 (2017)
13. Ordóñez Cabrera, M., Volodin, A.: Mean convergence theorems and weak laws of large numbers for weighted sums of random variables under a condition of weighted integrability. J. Math. Anal. Appl. **305**(2), 644–658 (2005)
14. Lita da Silva, J.: Convergence in p-mean for arrays of random variables. RM **74**(1), 1–11 (2019)
15. Sung, S.H.: Convergence in r-mean of weighted sums of NQD random variables. Appl. Math. Lett. **26**(1), 18–24 (2013)
16. Thành, L.V.: On the Baum-Katz theorem for sequences of pairwise independent random variables with regularly varying normalizing constants. Comptes Rendus Mathématique. Académie des Sciences. Paris **358**(11–12), 1231–1238 (2020)
17. Thành, L.V.: Mean convergence theorems for arrays of dependent random variables with applications to dependent bootstrap and non-homogeneous Markov chains. Stat. Pap. https://doi.org/10.1007/s00362-023-01427-y pp. 1–28 (2023)
18. Thành, L.V.: On a new concept of stochastic domination and the laws of large numbers. Test **32**(1), 74–106 (2023)
19. Thành, L.V.: On an extension of the Pyke–Root theorem. Manuscript pp. 1–16 (2023)
20. Thành, L.V.: On Rio's proof of limit theorems for dependent random fields. Manuscript pp. 1–27 (2023)
21. Thành, L.V.: The Hsu–Robbins–Erdös theorem for the maximum partial sums of quadruplewise independent random variables. J. Math. Anal. Appl. **521**(1), 126896 (2023)
22. Wu, Y., Guan, M.: Mean convergence theorems and weak laws of large numbers for weighted sums of dependent random variables. J. Math. Anal. Appl. **377**(2), 613–623 (2011)
23. Wu, Y., Wang, X.: Equivalent conditions of complete moment and integral convergence for a class of dependent random variables.Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Ser. A Math. **112**(2), 575–592 (2018)
24. Wu, Y., Wang, X., Hu, S., Yang, L.: Weighted version of strong law of large numbers for a class of random variables and its applications. Test **27**(2), 379–406 (2018)
25. Wu, Y., Wang, X., Hu, T.C., Ordóñez Cabrera, M., Volodin, A.: Limiting behaviour for arrays of rowwise widely orthant dependent random variables under conditions of R-h-integrability and its applications. Stochastics **91**(6), 916–944 (2019)